ISRAEL JOURNAL OF MATHEMATICS, Vol. 51, Nos. 1-2, 1985

SOLUBILITY OF FINITE GROUPS ADMITTING A FIXED-POINT-FREE AUTOMORPHISM OF ORDER *rst* III

BY

PETER ROWLEY

Department of Mathematics, UMIST Manchester M60 1QD, England

ABSTRACT

This paper, which is one of a series of four, contributes to the proof of the following

THEOREM. *A finite group admitting a coprime fixed-point-free automorphism* α *of order rst* (r, s and t distinct primes) *is soluble.*

Here we prove that in a minimal counterexample to the above theorem the set of α-invariant Sylow p-subgroups P, such that $C_P(\alpha^i) \neq 1$ for all $\alpha^i \neq 1$, generate a soluble subgroup.

10. Introduction

This paper is the third in a four-part series, whose aim is the proof of [Theorem 1.1; 3]. Our section numbering continues that of [3] and [4]. We recall from section 3 that $L_0 = \langle P \mid P$ α-invariant Sylow subgroup of G of type $\Psi = \{1, 2, 3\}\rangle$. As stated in section 1, Part III is concerned with proving, under the assumption that $(G, \langle \alpha \rangle)$ satisfies Hypothesis III, the following result.

THEOREM 10.1. *L_0 is an α-invariant soluble subgroup of G.*

By (2.4) this is equivalent to showing that every pair of α-invariant Sylow subgroups of G of type Ψ permute. Hence the proof of Theorem 10.1 amounts to showing that the existence of α-invariant Sylow subgroups of n-p type I, II, III, or IV in G leads to contradictory situations (for the definition of n-p type see [Definition 7.11; 4]).

In the remainder of this section we collect together certain observations concerning α-invariant Sylow subgroups of various n-p types. In section 11, 12

Received November 15, 1983 and in revised form October 30, 1984

and 13 we examine, respectively, the cases when G possesses only α-invariant Sylow subgroups of n-p types I, II and V; only types III, IV and V; and finally, all possible n-p types.

We continue to denote the α-invariant Sylow 2-subgroup of G by T. During this section P and Q will denote (respectively) α-invariant Sylow p- and q-subgroups of G of type Ψ.

LEMMA 10.2. *Let Q be of n-p type II. Then Q cannot, additionally, be of n-p type I, III or IV.*

PROOF. Let P be an α-invariant Sylow p-subgroup of G of n-p type I with respect to Q. So $\mathfrak{M}(p,q)=\{P, N_P(Q)Q\}$. By Lemma 7.9, without loss of generality, $Z(P)=Z(P)_{\sigma\tau}\leqq N_P(Q)$, and so $Q_{\sigma\tau}=1$. If, furthermore, Q was either of n-p type I or n-p type III, then, from Lemmas 7.9 and 7.10, $Z(Q)\leqq Q_{\alpha_i\alpha_j}$ where $i,j\in\Psi$, $i\neq j$. But then $[Z(Q),N_P(Q)]=1$ by (2.3)(xi) which contravenes the form of $\mathfrak{M}(p,q)$. Hence, we infer, Q cannot be of n-p type I or III. If Q were additionally of n-p type IV, then, by Lemma 7.10, $Q^*=Q_{\alpha_i}$ some $i\in\Psi$. Since $Q_{\sigma\tau}=1$, this implies $\alpha_i=\rho$, against Lemma 7.9(e). This finishes the proof of the lemma.

LEMMA 10.3. *Suppose P is of n-p type I with respect to Q. Let M be a non-trivial α-invariant nilpotent Hall μ-subgroup of G with $(\mu,p)=(\mu,q)=1$. If $PM=MP$ and $QM=MQ$, then*

(i) $O_p(PM)=C_{O_p(PM)}(M)(O_p(PM)\cap N_P(Q))$; *and*

(ii) *either* $N_Q(M_1)=1$ *for all non-trivial α-invariant subgroups M_1 of M or* $O_p(PM)\leqq N_P(Q)$ *and* $p=2$.

PROOF. From Lemma 7.9 $\mathfrak{M}(p,q)=\{P,N_P(Q)Q\}$ with $P^*\leqq N_P(Q)$.

(i) From Lemma 5.8(d) (i) holds provided $Q(PM)\neq G$. So we may suppose $Q(PM)=G$. Now $O_\mu(QM)^{(QM)P}=O_\mu(QM)^P\leqq PM\neq G$ and Hypothesis III forces $Q_\mu(QM)=1$. Hence $M=N_M(J(Q))C_M(Z(Q))$ by (2.6). Clearly $N_P(Q)$ permutes with both $N_M(J(Q))$ and $N_M(Z(Q))$ and consequently $N_P(Q)$ permutes with $\langle N_M(J(Q)),N_M(Z(Q))\rangle=M$. Using (2.14)(ii) now yields the desired conclusion.

(ii) If $N_Q(M_1)\neq 1$ for some non-trivial α-invariant subgroup M_1 of M, then (i) implies, because of the shape of $\mathfrak{M}(p,q)$, that $O_p(PM)\leqq N_P(Q)$. Hence we also have $p=2$. For otherwise, by Corollary 4.5, $P=O_p(PM)P^*\leqq N_P(Q)$, contrary to $PQ\neq QP$. This verifies (ii).

LEMMA 10.4. *Suppose P is of n-p type I with respect to Q. If W is an*

α-invariant Sylow w-subgroup of G of type Ψ, $p \neq w$ and $PW = WP$, then $WQ = QW$, $O_p(PW) \leqq N_P(Q)$ and $p = 2$.

PROOF. By Lemmas 3.14 and 10.3(ii) it suffices to show that WQ = QW. Suppose $WQ \neq QW$ and argue for a contradiction. Combining Lemmas 7.8 and 10.2 we see that W must be of n-p type I with respect to Q. Thus $\mathfrak{M}(p,q) = \{P, N_P(Q)Q\}$ and $\mathfrak{M}(w,q) = \{W, N_W(Q)Q\}$ with $P^* \leqq N_P(Q)$ and $W^* \leqq N_W(Q)$. We may suppose that (say) $2 \neq w$. Using (2.14)(ii) and Corollary 4.5 we may deduce that $W = C_W(N_P(Q))N_W(Q)$. But then the shape of $\mathfrak{M}(w,q)$ and Lemma 3.14 force $W \leqq N_W(Q)$, which is the required contradiction.

LEMMA 10.5. *Suppose T (the α-invariant Sylow 2-subgroup of G) is of type Ψ. If T is of n-p type III with respect to P, then P cannot also be of n-p type I.*

PROOF. We shall show that P being of n-p type I leads to a contradiction. Set $\mathfrak{M}(2,p) = \{TY, XP\}$. Without loss of generality we may suppose that $T_\rho \leqq X$ and P_σ, $P_\tau \leqq Y$. Hence, from Lemma 7.10, $Y \leqq P_\rho = P^* \neq P$, and $Z(T) = Z(T)_{\sigma\tau} \leqq X$.

Suppose P is of n-p type I with respect to (say) Q. Thus $P^* \leqq N_P(Q)$ and $\mathfrak{M}(p,q) = \{P, N_P(Q)Q\}$. Clearly $T \neq Q$.

First we consider the possibility that $TQ \neq QT$. Since Q is of n-p type II, Q cannot be of n-p type I, III or IV by Lemma 10.2. In particular, T must be of n-p type I with respect to Q, and so P^*, $T^* \leqq N_G(Q)$. But then one of $P^* \leqq Y$ and $T^* \leqq X$ must hold which is impossible. Therefore we have $TQ = QT$. Since $P^* \nleqq Y$, $N_T(J(Q))$, $C_T(Z(Q)) \leqq X$ and so $T = O_2(TQ)X$ by (2.6). Suppose that $(O_2(TQ))_\rho = 1$. Thus, using (2.11), $[O_2(TQ), [Q,\rho]] = 1$. By Lemma 7.9(f) $[Q,\rho] \neq 1$ and so, employing (2.3)(viii),

$$O_2(TQ), P_\rho \leqq (N_G([Q,\rho]))_{\{2,p\}}.$$

Since $O_2(TQ) \nleqq X$ and $P_\rho = P^* \nleqq Y$, we conclude that $(O_2(TQ))_\rho \neq 1$. Hence, by Lemma 7.10(c), $Z(O_2(TQ)) \leqq X$. Set $Z(O_2(TQ)) = Z$. Note that $Z \neq 1$. Applying Lemma 5.1(a) (since $P_\rho, P_\tau \leqq Y$) yields $P = YC_P([Z, \sigma\tau])$. Since neither $P = Y$ nor $O_2(TQ) \leqq X$ is permissible, $Z = Z_{\sigma\tau}$. Consequently $[Z, Q] = 1$ by (2.3)(xi). Considering $(N_G(Q))_{\{2,p\}}$ we see that Z normalizes $O_p(PX) \cap N_P(Q)$ ($\geqq (O_p(PX))^*$). So, by (2.14)(ii),

$$O_p(PX) = C_{O_p(PX)}(Z)(O_p(PX) \cap N_P(Q)).$$

Because $O_2(TQ) \nleqq X$, $C_P(Z) \leqq Y \leqq P_\rho \leqq N_P(Q)$, and so, using (2.20), $P = YO_p(PX) \leqq N_P(Q)$, whereas $PQ \neq QP$. This is the desired contradiction, so completing the proof.

In Lemmas 10.6 and 10.7 we assume P to be of n-p type I with respect to Q, and, by Lemma 7.9, may suppose $Z(P) \leqq P_{\sigma\tau}$ and (hence) $Q_{\sigma\tau} = 1$.

LEMMA 10.6. (i) *P permutes with $\mathscr{L}_1$.*

(ii) *Q permutes with L_{23}.*

(iii) $L_{12} = L_{13} = 1$.

(iv) *Q permutes with at least one of L_2 and L_3.*

(v) *If $L_iP = PL_i$ and $L_iQ = QL_i$ where $i = 2$ or 3, then $L_i = 1$.*

(vi) *If $L_{23} \neq 1$, then $PL_{23} \neq L_{23}P$.*

PROOF. (i) From Lemma 3.13(iii) $[P_{\sigma\tau}, \mathscr{L}_1] = 1$. Hence $P, \mathscr{L}_1 \leqq C_G(Z(P))$, so giving $P\mathscr{L}_1 = \mathscr{L}_1P$.

(ii) Since $[Q_\rho, L_{23}] = 1$ by Lemma 3.6(iii), $\mathscr{P}_Q(L_{23}) \neq 1$. Hence, because $\mathscr{P}_Q(L_{23})L_{23}$ admits $\sigma\tau$ fixed-point-freely, $QL_{23} = L_{23}Q$ by (2.8) and (2.21)(v).

(iii) We show that $L_{12} = 1$; a similar argument gives $L_{13} = 1$. First we demonstrate that $QL_{12} = L_{12}Q$. Suppose $QL_{12} \neq L_{12}Q$. By Lemma 3.6(iii) $[Q_\tau, L_{12}] = 1$. Thus $Z(Q) \leqq \mathscr{P}_Q(L_{12})$ and so $Z(Q) \leqq Q_\tau$ by Lemma 5.1(b). Lemma 7.9(e) forbids such a situation, and so $QL_{12} = L_{12}Q$. Since $[Q_\tau, L_{12}] = 1$, $O_{\pi\widehat{12}}(PL_{12}) = 1$ because of the shape of $\mathfrak{M}(p, q)$. Observe, as $Z(P) \leqq Z(PL_{12})$ and $Z(P) \leqq N_P(Q)$, that $G \neq Q(PL_{12})$, and therefore $p = 2$ by Lemma 5.8(e)(ii). Employing Corollary 4.5 (for PL_{12}) gives $L_{12} \leqq G_\tau$, which, by (2.8) and Lemma 6.1, is contrary to $(G, \langle \alpha \rangle)$ satisfying Hypothesis III unless $L_{12} = 1$.

(iv) Since $\sigma\tau$ acts fixed-point-freely upon α-invariant $\{q \cup \pi_2 \cup \pi_3\}$-subgroups of G, $[L_2, Q_\tau] = [L_3, Q_\sigma] = 1$ by Lemma 3.6(ii). Therefore $L_2Q \neq QL_2$ and $L_3Q \neq QL_3$ would give $Z(Q) \leqq Q_{\sigma\tau}$ by Lemma 5.1(b), contrary to $Q_{\sigma\tau} = 1$. Hence (iv) holds.

(v) Suppose $L_i \neq 1$ and take $i = 2$. Then, since $[L_2, Q_\tau] = 1$, $P \leqq N_P(Q)$ by (2.7), the form of $\mathfrak{M}(p, q)$ and Lemma 10.3(ii). Therefore $L_2 = 1$.

(vi) This may be established by arguing as in (iii).

LEMMA 10.7. *If $L_1Q \neq QL_1$, then*

(i) $2 \in \pi_1$; *and*

(ii) $L_1^* \leqq N_{L_1}(Q)$.

PROOF. We first demonstrate that $L_1^* \leqq N_{L_1}(Q)$ pertains. Assume $L_1^* \not\leqq N_{L_1}(Q)$. Thus, without loss of generality, $L_{1_\sigma} \not\leqq \mathscr{P}_{L_1}(Q)$. Note that $Z(Q) \leqq \mathscr{P}_Q(L_1)$ is untenable here. For $Z(Q) \leqq \mathscr{P}_Q(L_1)$ implies $Z(Q) \leqq Q_\rho$ contrary to Lemma 7.9(e). Thus $O_q(Q_\sigma L_{1_\sigma}) = 1$. Hence $Z(Q_\sigma) \leqq Q_\rho$ by (2.13)(iii) which gives the untenable $[Z(Q_\sigma), P_\sigma] = 1$ (by (2.3)(xi)). Therefore $L_1^* \leqq N_{L_1}(Q)$. Combining

(2.14)(ii) and Lemma 3.14 with the fact that $\mathcal{P}_O(L_1) = 1$ yields $O_{\pi_1}(PL_1) = 1$ (P and L_1 permute by Lemma 10.6(i)). Consequently $2 \in \pi_1$ by Corollary 4.5.

For Lemmas 10.8 and 10.9 we assume T is of type Ψ and of n-p type III with respect to Q. Set $\mathfrak{M}(2, q) = \{TY, QX\}$, and suppose $T_\rho \leqq X$ and Q_σ, $Q_\tau \leqq Y$ pertains.

LEMMA 10.8. (i) $T\mathcal{L}_1 = \mathcal{L}_1 T$.

(ii) *Q permutes with L_{12}, L_{13} and L_{23}.*

(iii) $L_{12} = L_{13} = 1$.

(iv) *Q permutes with at least one of L_2 and L_3.*

(v) *If $L_{23} \neq 1$, then $L_{23}T \neq TL_{23}$.*

PROOF. (i) From Lemma 7.10(d) $Z(T) \leqq T_{\sigma\tau}$ and so $(C_G(Z(T)))_{\{2,\pi_1\}} \geqq T, \mathcal{L}_1$ by Lemma 3.13(iii). So (i) holds.

(ii) Since $Q_{\sigma\tau} = 1$ (by Lemma 7.10), $(L_{23})^*_{\langle\sigma\tau\rangle} = 1$ and $1 \neq Q_\rho \leqq C_O(L_{23}) \leqq \mathcal{P}_O(L_{23})$, (2.8) and (2.21)(v) yields $QL_{23} = L_{23}Q$.

Next we show that $QL_{12} = L_{12}Q$. Suppose that $QL_{12} \neq L_{12}Q$, and argue for a contradiction. Now $1 \neq Q_\tau \leqq C_O(L_{12})$ (by Lemma 3.13(iii)) and $\mathfrak{M}(\pi_{12}, q) = \{Q, L_{12}N_Q(L_{12})\}$ with, since $q \neq 2$, $N_Q(L_{12}) = C_O(L_{12})(N_Q(L_{12}))_\rho$ by (2.11). Since $L_{12} \neq 1$ and $(G, \langle\alpha\rangle)$ satisfies Hypothesis III, combining (2.8) and Lemma 6.1, we have $L_{12} \neq (L_{12})_\tau = (L_{12})^*$. From (i) we have $TL_{12} = L_{12}T$. Therefore Corollary 4.5 dictates that $O_{\pi_{12}}(TL_{12}) \neq 1$ (clearly $2 \notin \pi_{12}$). Considering

$$(N_G(O_{\pi_{12}}(TL_{12})))_{\{2,q\}} \geqq C_O(L_{12}), T$$

we obtain $C_O(L_{12}) \leqq Y \leqq Q_\rho$. So $N_O(L_{12}) = C_O(L_{12})(N_O(L_{12}))_\rho \leqq Q_\rho$. Since $C_O(N_O(L_{12})) \leqq N_O(L_{12})$, (2.3)(xii) forces $Q = Q_\rho$ which contradicts Lemma 7.10(b).

Therefore we conclude that $L_{12}Q = QL_{12}$. That $L_{13}Q = QL_{13}$ follows by similar reasoning.

(iii) Suppose $L_{12} \neq 1$. By Lemma 6.1 $L_{12} \neq L_{12_\tau}$. From parts (i) and (ii) $TL_{12} = L_{12}T$ and $QL_{12} = L_{12}Q$. Thus $O_{\pi_{12}}(TL_{12}) \neq 1$ by Corollary 4.5 and hence $T(L_{12}Q) \neq G$. Now Lemma 5.8(f) (since $Q_\sigma \leqq Y$ and $L_{12_\sigma} = 1$) predicts that either $O_{\pi_{12}}(TL_{12}) = 1$ or $q = 2$. Since neither possibility holds, we have a contradiction. Therefore $L_{12} = 1$; again similar arguing yields $L_{13} = 1$.

For parts (iv) and (v) see Lemma 10.6(iv) and (v).

LEMMA 10.9. *If $TL_i \neq L_iT$ where $i = 2$ or 3, then $L_iQ = QL_i$.*

PROOF. Suppose the lemma is false and argue for a contradiction. Without loss of generality set $i = 2$.

Since $Q_{\sigma\tau} = 1 = (L_2)_{\sigma\tau}$ and $[L_{2_\tau}, Q_\tau] = 1$, Lemma 3.6(ii) implies $[L_2, Q_\tau] = 1$. Hence $\mathfrak{M}(\pi_2, q) = \{L_2N_Q(L_2), Q\}$ by Lemma 5.3 and (2.19). So Q_ρ, $Q_\tau \leqq N_Q(L_2)$ whence $Q^* \leqq N_Q(L_2)$.

Since, by hypothesis, $L_2T \neq TL_2$ recourse to Corollary 7.4 yields the four following possible situations.

(i) $L_2^* \leqq \mathcal{P}_{L_2}(T)$;

(ii) $T_\tau \leqq \mathcal{P}_T(L_2)$ and $L_{2_\rho} \leqq \mathcal{P}_{L_2}(T)$;

(iii) $T_\rho \leqq \mathcal{P}_T(L_2)$ and $L_{2_\tau} \leqq \mathcal{P}_{L_2}(T)$; and

(iv) $T^*_{\langle \rho\tau\rangle} \leqq \mathcal{P}_T(L_2)$.

We consider each case in turn.

(i) Here (see Lemma 7.5) $\mathcal{P}_{L_2}(T) = N_{L_2}(T)$. From Lemma 7.10 $Z(T) \leqq T_{\sigma\tau}$ and consequently $[Z(T), N_{L_2}(T)] = 1$ by (2.3)(xi). This produces the impossible $1 \neq Z(T) \leqq \mathcal{P}_T(L_2) = 1$.

(ii) Now $Z(T) \leqq T_{\sigma\tau} \leqq T_\tau \leqq \mathcal{P}_T(L_2)$. Hence, as $L_{2_\rho} \leqq \mathcal{P}_{L_2}(T)$, appealing to Lemma 5.1(a) gives $L_2 = \mathcal{P}_{L_2}(T)C_{L_2}(Z(T))$ whence $L_2 = \mathcal{P}_{L_2}(T)$, a contradiction.

(iii) Since $Q^* \leqq N_Q(L_2)$ and $Q^* \nleqq Y$, we have, with the aid of (2.7),

$$\mathcal{P}_T(L_2) = (\mathcal{P}_T(L_2) \cap X)O_2(\mathcal{P}_T(L_2)L_2).$$

We consider two cases depending on whether $O_2(\mathcal{P}_T(L_2)L_2)$ is trivial or non-trivial. First suppose $O_2(\mathcal{P}_T(L_2)L_2) = 1$. Then $\mathcal{P}_T(L_2) \leqq X$. Clearly $L_2(XQ) \neq G$ and so $\mathcal{P}_{XQ}(L_2) = \mathcal{P}_X(L_2)\mathcal{P}_Q(L_2) = \mathcal{P}_T(L_2)N_Q(L_2)$. Now $Q^* \leqq N_Q(L_2)$ and, by Lemma 7.7(b), $Z(T) \leqq \mathcal{P}_T(L_2)$. Therefore, using (2.14)(ii), we have

$$\begin{aligned} O_q(QX) &= (O_q(QX) \cap N_Q(L_2))C_{O_q(QX)}(Z(T)) \\ &\leqq N_Q(L_2), \end{aligned}$$

since $C_Q(Z(T)) \leqq Y \leqq Q_\rho \leqq N_Q(L_2)$. But then $Q = Q^*O_q(QX) \leqq N_Q(L_2)$, a contradiction.

So now we consider the possibility $O_2(\mathcal{P}_T(L_2)L_2) \neq 1$. Hence (see (2.21)(ii))

$$Z = Z(O_2(T\mathcal{P}_{L_2}(T))) \leqq C_T(O_2(\mathcal{P}_T(L_2)L_2)) \leqq \mathcal{P}_T(L_2).$$

Because $L_{2_\sigma} = 1$, by (2.12), $[Z, \sigma]$ centralizes $O_{\pi_2}(\mathcal{P}_T(L_2)L_2)$. Neither $O_2(T\mathcal{P}_{L_2}(T)) \leqq \mathcal{P}_T(L_2)$ nor $O_{\pi_2}(\mathcal{P}_T(L_2)L_2) \leqq \mathcal{P}_{L_2}(T)$ can hold and so $Z \leqq T_\sigma$. Applying (2.3)(xi) we obtain $[Z, \mathcal{P}_{L_2}(T)] = 1$. Therefore $Z \leqq N_T(L_2)$ by (2.20). By considering $(N_G(L_2))_{\{2,q\}}$ and imitating the arguments for the case $O_2(\mathcal{P}_T(L_2)L_2) = 1$ with Z in place of $Z(T)$ we obtain the contradiction $Q = N_Q(L_2)$.

(iv) Since $Z(T) \leqq T_{\sigma\tau} \leqq T^*_{\langle\rho\tau\rangle} \leqq \mathcal{P}_T(L_2)$, this case may be dealt with as in (iii).
Thus we have shown that $L_iT \neq TL_i$ ($i = 2$ or 3) implies that $L_iQ = QL_i$.

11. α-Invariant Sylow subgroups of n-p type I and II

We introduce the following hypothesis.

HYPOTHESIS 11.1. G possesses α-invariant Sylow subgroups of n-p type I and II but none of n-p type III and IV.

THEOREM 11.2. *Hypothesis* 11.1 *is incompatible with Hypothesis III.*

We present the proof of Theorem 11.2 in a series of lemmas. The incompatibility of the two hypotheses will follow from a particular factorization of G. The road to this factorization will be paved by Lemmas 11.3 to 11.6, during which we assume P and Q to be α-invariant Sylow subgroups of type Ψ with P of n-p type I with respect to Q. Also we assume $Z(P) \leqq P_{\sigma\tau}$ holds; hence $Q_{\sigma\tau} = 1$.

LEMMA 11.3. *Suppose $PL_i = L_iP$ for some $i \in \Psi$, and let W be an α-invariant Sylow w-subgroup of type Ψ which permutes with P. Then $WL_i = L_iW$.*

PROOF. Supposing $WL_i \neq L_iW$ we shall deduce a contradiction. Thus $p \neq w$. From Lemma 10.4 we have $WQ = QW$, $O_p(PW) \leqq N_P(Q)$ and $p = 2$.

If $L_i^* \leqq N_{L_i}(W)$ were to hold, then, since $2 \notin \pi_i$, Lemmas 5.8(e)(ii) and 7.5(a) imply that $N_W(J) = 1$ for all non-trivial α-invariant subgroups J of P. But this is contrary to Lemma 3.14, and therefore $L_i^* \not\leqq N_{L_i}(W)$.

Now suppose $i = 1$. Consequently, since $p = 2$, $W^*_{\langle\sigma\tau\rangle} \leqq N_W(L_1)$ by Corollary 7.4 and Lemma 7.6. Moreover, from Lemma 10.7 we also have $QL_1 = L_1Q$. Since $Q_{\sigma\tau} = 1$, applying Lemma 5.8(f) to Q, L_1 and W (note $(QW)L_1 \neq G$) yields that $O_q(QL_1) = 1$. Hence $Q = Q_\rho$ by (2.13)(i). Lemma 7.9(f) forbids such a situation, and so we have verified the lemma when $i = 1$.

We now consider the case $i = 2$. Since $L_2^* \not\leqq N_{L_2}(W)$, $W^*_{\langle\rho\tau\rangle} \leqq \mathcal{P}_W(L_2) = N_W(L_2)$ by Corollary 7.4. By Lemma 3.7, because all α-invariant $\{q \cup \pi_2\}$-subgroups of G admit $\sigma\tau$ fixed-point-freely, $[L_2, Q_\tau] = 1$. Then the shape of $\mathfrak{M}(p, q)$ dictates that $N_P(J) \leqq N_P(Q)$ for all non-trivial α-invariant subgroups J of L_2. Hence, by (2.7), $P = O_p(PL_2)N_P(Q)$, and therefore $O_p(PL_2) \not\leqq N_P(Q)$. Employing (2.14)(ii) and (2.26) gives

$$O_p(PL_2) = C_{O_p(PL_2)}(\mathcal{P}_{L_2}(Q))(O_p(PL_2) \cap N_P(Q)).$$

Consequently $\mathcal{P}_{L_2}(Q) = 1$, and so, in particular, $QL_2 \neq L_2Q$. Therefore, by Corollary 7.4, $Q^*_{\langle\rho\tau\rangle} \leqq \mathcal{P}_Q(L_2) = N_Q(L_2)$.

It is asserted that either $Q_\tau \leqq Q_\rho$ or $O_w(WQ) \leqq N_W(L_2)$ holds. Clearly $N_Q(L_2)$ and $N_W(L_2)$ permute and so (since $W_\rho \leqq N_W(L_2)$ and $q \neq 2$)

$$O_w(WQ) = C_{O_w(WQ)}([N_Q(L_2), \rho])(O_w(WQ) \cap N_W(L_2))$$

by (2.14)(i). Recalling that $Q_\tau \leqq C_Q(L_2)$ yields the assertion. By Lemma 7.9(e) $Q_\tau \leqq Q_\rho$ cannot occur. Thus $O_w(WQ) \leqq N_W(L_2)$ which implies, by Corollary 4.5 and Lemmas 4.6 and 7.6(d), that $W = W_\sigma$. Hence $P = O_p(PW)P_\sigma$ by (2.3)(ix), and thus $P = O_p(PW)P_\sigma \leqq N_P(Q)P^* \leqq N_P(Q)$, which is untenable. This contradiction shows that $WL_2 = L_2W$.

The case $i = 3$ may be established by arguments analogous to those used for $i = 2$. The lemma now follows.

LEMMA 11.4. *Suppose L_i permutes with P, where $i = 2$, or 3. Then $L_iL_1 = L_1L_i$.*

PROOF. As usual we suppose the lemma is false, and argue for a contradiction. Without loss of generality we may set $i = 2$. Using Lemma 3.7 we have that $[L_2, Q_\tau] = 1$. Hence, because of the shape of $\mathfrak{M}(p, q)$,

$$O_{\pi_2}(PL_2) = 1. \tag{11.1}$$

Using (2.7), (2.14)(ii) and (2.26) (as in the proof of Lemma 11.5) gives $P = C_P(\mathcal{P}_{L_2}(Q))N_P(Q)$, whence (using (2.20))

$$QL_2 \neq L_2Q, \quad \mathfrak{M}(q, \pi_2) = \{N_Q(L_2)L_2, Q\} \quad \text{and} \quad Q^*_{\langle \rho\tau \rangle} \leqq N_Q(L_2). \tag{11.2}$$

If $QL_1 \neq L_1Q$, then, since $L_1L_2 \neq L_2L_1$ by hypothesis, Theorem 8.1 implies $QL_2 = L_2Q$, which is contrary to (11.2). Therefore

$$QL_1 = L_1Q. \tag{11.3}$$

By Lemma 5.1(b) $\mathcal{P}_{L_2}(L_1) \cap Z(L_2) \leqq L_{2_\rho}$. Therefore, if $L_{2_\tau} \leqq \mathcal{P}_{L_2}(L_1)$ holds, then $Z(L_2)^* = Z(L_2)_\rho$. Combining (11.1) and Corollary 4.5 then gives $Z(L_2) = Z(L_2)_\rho$. Since $C_Q(L_2) \neq 1$, $Z(Q) \leqq N_Q(L_2)$ and so, by (2.3)(x) and the shape of $\mathfrak{M}(q, \pi_2)$, $Z(Q) \leqq Q_\rho$. This is against Lemma 7.9(e), and thus $L_{2_\tau} \not\leqq \mathcal{P}_{L_2}(L_1)$. Hence $L_{1_\tau} \leqq \mathcal{P}_{L_1}(L_2)$.

Since $Q_\rho \leqq N_Q(L_2)$ and $L_{1_\rho} = 1$, (2.14)(ii) and (2.25) give

$$O_q(QL_1) = C_{O_q(QL_1)}(\mathcal{P}_{L_1}(L_2))(O_q(QL_1) \cap N_Q(L_2)).$$

$$\text{Either } O_q(QL_1) \leqq N_Q(L_2) \quad \text{or} \quad C_Q(\mathcal{P}_{L_1}(L_2)) \neq 1 = C_{L_2}(L_{1_\tau}). \tag{11.4}$$

If $C_Q(\mathcal{P}_{L_1}(L_2)) = 1$, then clearly $O_q(QL_1) \leqq N_Q(L_2)$. Since $L_{1_\tau} \leqq \mathcal{P}_{L_1}(L_2)$, $C_{L_2}(L_{1_\tau}) \neq 1$ and the shape of $\mathfrak{M}(q, \pi_2)$ would force $O_q(QL_1) \leqq N_Q(L_2)$.

Suppose $O_q(QL_1) \leqq N_Q(L_2)$ holds. Then $q = 2$ for otherwise, using (2.13)(i), $Q = O_q(QL_1)Q_\rho \leqq N_Q(L_2)$. Since $p \neq 2$ and, by Lemma 10.6, $PL_1 = L_1P$, Lemma 10.3(ii) implies that $N_Q(J) = 1$ for all non-trivial α-invariant subgroups J of L_1. Hence, using Corollary 4.5 on both QL_1 and $Q_\sigma L_{1_\sigma}$, we have $L_1 = L_1^* = L_{1_\tau}$, which contradicts $L_{1_\tau} \leqq \mathcal{P}_{L_1}(L_2) \neq L_1$. Consequently $O_q(QL_1) \not\leqq N_Q(L_2)$.

So, from (11.4), $C_Q(\mathcal{P}_{L_1}(L_2)) \neq 1$ and hence, by Lemma 10.3(ii), $p = 2$. Consequently L_2 is star-covered by (11.1) and Corollary 4.5, and $\mathcal{P}_{L_1}(L_2) = N_{L_1}(L_2)$ by Lemma 5.7. Hence, by (2.3)(xi) and (11.4), $L_2/\Phi(L_2) = \bar{L}_2 = (\bar{L}_2)_\rho(\bar{L}_2)_\tau = C_{\bar{L}_2}(L_{1_\tau})(\bar{L}_2)_\tau = (\bar{L}_2)_\tau$. Thus $L_2 = L_{2_\tau}$. Since $Z(Q) \leqq N_Q(L_2)$ and $L_{2_\sigma} = 1$, we now obtain $Z(Q) = Z(Q)_{\sigma\tau}$ by (2.3)(x).

This contradictory state of affairs concludes the proof of Lemma 11.4.

LEMMA 11.5. *If $PL_i \neq L_iP$ where $i = 2$, or 3, then $QL_i = L_iQ$.*

PROOF. Deny the result and, without loss of generality, suppose $i = 2$. Thus $PL_2 \neq L_2P$ and $QL_2 \neq L_2Q$. Observe that $L_2^* \leqq \mathcal{P}_{L_2}(P)$ is impossible. For $L_2^* \leqq \mathcal{P}_{L_2}(P)$ implies that $\mathfrak{M}(p, \pi_2) = \{N_{L_2}(P)P, L_2\}$ by Lemma 7.5 whereas $Z(P) \leqq P_{\sigma\tau}$ gives, using (2.3)(xi), that $Z(P) \leqq C_P(N_{L_2}(P)) \leqq \mathcal{P}_P(L_2)$.

We now show that $Z(P) \leqq \mathcal{P}_P(L_2)$. If $P_\tau \leqq \mathcal{P}_P(L_2)$, then clearly we have $Z(P) \leqq \mathcal{P}_P(L_2)$. So, since $L_2^* \not\leqq \mathcal{P}_{L_2}(P)$, we may assume $P_\rho \leqq \mathcal{P}_P(L_2)$ and $L_{2_\tau} \leqq \mathcal{P}_{L_2}(P)$ in which case $Z(P) \leqq \mathcal{P}_P(L_2)$ by Lemma 7.7(b).

Since $O_{\pi_2}(L_2\mathcal{P}_P(L_2)) \neq 1$ and, by Lemma 3.7, $[L_2, Q_\tau] = 1$, by considering $(N_G(O_{\pi_2}(L_2\mathcal{P}_P(L_2))))_{\{p,q\}}$ together with the form of $\mathfrak{M}(p, q)$ we obtain $\mathcal{P}_P(L_2) \leqq N_P(Q)$. If $Q_\rho \leqq \mathcal{P}_Q(L_2)$, then, since $Z(P)_\rho = 1$ and $\mathcal{P}_Q(P) = 1 \leqq \mathcal{P}_Q(L_2)$, Lemma 5.10(c) (with $L = P$, $M = Q$ and $N = L_2$) predicts that $\mathcal{P}_Z(Q) \cap Z(P) = 1$ where $Z = \mathcal{P}_P(L_2)$. However $Z(P) \leqq \mathcal{P}_P(L_2) \leqq N_P(Q)$ and so $1 \neq Z(P) \leqq \mathcal{P}_Z(Q) \cap Z(P)$. Hence we must have $L_{2_\rho} \leqq \mathcal{P}_{L_2}(Q)$. Since $1 \neq L_{2_\rho} \leqq \mathcal{P}_{L_2}(Q)$ and $1 \neq Q_\tau \leqq C_Q(L_2)$, $q = 2$ by Lemma 5.3. Hence, by Corollary 7.4 and Lemmas 7.5 and 7.6, $\mathcal{P}_{L_2}(Q) = N_{L_2}(Q)$, $\mathcal{P}_P(L_2) = N_P(L_2)$ and $\mathcal{P}_{L_2}(P) \leqq L_{2_{\rho\tau}}$. Therefore $Z(P) \leqq N_P(L_2) = N_P(L_2) \cap N_P(Q) = \mathcal{P}_{N_P(Q)}(L_2)$. Since $\mathcal{P}_{L_2}(P) \leqq L_{2_{\rho\tau}} \leqq \mathcal{P}_{L_2}(Q)$, Lemma 5.10(c) (with $L = P$, $M = L_2$ and $N = Q$) again gives a contradiction.

This completes the proof of Lemma 11.5.

LEMMA 11.6. *If Q_1 and Q_2 are α-invariant Sylow subgroups of G both of n-p type II with respect to P, then $Q_1Q_2 = Q_2Q_1$.*

PROOF. If $Q_1Q_2 \neq Q_2Q_1$, then by Hypothesis 11.1 one of Q_1 and Q_2 must be of n-p type I, contrary to Lemma 10.2. Thus $Q_1Q_2 = Q_2Q_1$.

For P an α-invariant Sylow p-subgroup of G of type Ψ we introduce the following notation:

$$H_0(p) = \langle W \mid W \text{ is an } \alpha\text{-invariant Sylow subgroup of type } \Psi \text{ and } WP = PW \rangle,$$

$$K_0(p) = \langle W \mid W \text{ is an } \alpha\text{-invariant Sylow subgroup of type } \Psi \text{ and } WP \neq PW \rangle.$$

Also we set $\pi(H_0(p)) = \pi_0(p)$.

LEMMA 11.7. *If $H_0(p)$ is not an α-invariant soluble Hall $\pi_0(p)$-subgroup of G, then*

(i) $p = 2$; *and*

(ii) *there exists an α-invariant Sylow w-subgroup of G of n-p type I with $w \neq 2$.*

PROOF. By (2.4) there must exist α-invariant Sylow u- and v-subgroups U and V (respectively) both of type Ψ with $UV \neq VU$, $UP = PU$ and $VP = PV$. Note that $U \neq P \neq V$. We may suppose that U is of n-p type I with respect to V. So $U^* \leqq \mathcal{P}_U(V) = N_U(V)$.

Suppose $p \neq 2$. Then Lemma 5.8(f) implies that $O_p(PV) = 1$. Hence $P = P^*$ by Theorem 4.4, contrary to P being of n-p type I. Thus $p = 2$, so giving (i). Taking $W = U$ yields (ii).

COROLLARY 11.8. *There exists an α-invariant Sylow p-subgroup P of G of type Ψ such that $H_0(p)$ is a soluble α-invariant Hall $\pi_0(p)$-subgroup of G.*

PROOF. By Hypothesis 11.1 there exists at least one α-invariant Sylow p-subgroup, P, of n-p type I. If $H_0(p)$ is not a soluble Hall $\pi_0(p)$-subgroup of G then, by Lemma 11.7(ii), there exists an α-invariant Sylow w-subgroup of G of n-p type I with $w \neq 2$ and so, by Lemma 11.7(i), $H_0(w)$ is an α-invariant soluble Hall $\pi_0(w)$-subgroup of G. The corollary now follows.

We are now in a position to factorize G. For the remainder of this section P will denote an α-invariant Sylow p-subgroup of G of type Ψ such that $H_0(p)$ is a soluble Hall subgroup of G. Further, H will denote the subgroup of G generated by $H_0(p)$ and those of $\{L_i, L_{jk} \mid i, j, k \in \Psi\}$ which permute with P. Let K denote the subgroup of G generated by $K_0(p)$ and those of $\{L_i, L_{jk} \mid i, j, k \in \Psi\}$ which do not permute with P. For $H_0(p)$ and $K_0(p)$ we will now write (respectively) H_0 and K_0.

LEMMA 11.9. *$G = HK$ with H and K α-invariant soluble Hall subgroups of G.*

PROOF. From Lemma 10.6 we recall that $L_{12} = L_{13} = 1$, that $PL_1 = L_1P$ and that, if $L_{23} \neq 1$, $PL_{23} \neq L_{23}P$. Thus, if $L_{23} \neq 1$, then $L_{23} \leqq K$. By Lemma 10.6(ii) L_{23} permutes with K_0. From Lemma 11.6 and (2.4) K_0 is a soluble Hall subgroup of G, and, using (2.4), (2.5) and Lemma 11.3, $H_0L_1 = L_1H_0$ is a soluble Hall subgroup of G.

Suppose both L_2 and L_3 do not permute with P. Then $L_2L_3 = L_3L_2$ by Theorem 8.1. Further (2.4), (2.5) and Lemma 11.5 yield that $L_2K_0 = K_0L_2$ and $L_3K_0 = K_0L_3$ are soluble Hall subgroups of G. Since $[L_2, L_{23}] = [L_3, L_{23}] = 1$, we have $G = HK$ with H and K α-invariant soluble Hall subgroups of G.

Now suppose that (say) L_2 permutes with P but L_3 does not permute with P. Using Lemmas 11.3 and 11.4 we may infer that $L_2L_1H_0$ is a soluble Hall subgroup of G. From Lemma 11.5 we also have that L_3K_0 is a soluble Hall subgroup of G. Therefore, in this situation, the lemma holds.

Now we consider the case when both L_2 and L_3 permute with P. Then, by Lemma 10.6(iv) and (v), one of L_2 and L_3 must be trivial, and so Lemmas 11.3, 11.4 and 11.5 give the desired factorization.

This exhausts all the possibilities, and so Lemma 11.9 is verified.

The next two lemmas will be used to show that the factorization obtained in Lemma 11.9 contradicts Hypothesis III. In these two lemmas Q denotes a (non-trivial) α-invariant Sylow q-subgroup of K_0.

LEMMA 11.10. (i) *Suppose* $PL_i \neq L_iP$ (*where* $i = 2$ *or* 3) *and* $q \neq 2$. *Then* $\mathfrak{M}(p, \pi_i) = \{P, N_P(L_i)L_i\}$.

(ii) *If* $PL_{23} \neq L_{23}P$ *and* $q \neq 2$, *then* $\mathfrak{M}(p, \pi_{23}) = \{P, N_P(L_{23})L_{23}\}$.

PROOF. (i) In view of (2.20) it will be sufficient to show that $\mathcal{P}_{L_i}(P) = Y = 1$. Suppose $Y \neq 1$, and argue for a contradiction. By Lemma 5.1(d) and (2.21)(vi) $P_{\alpha_i} \not\leqq \mathcal{P}_P(L_i)$. Consequently, as $P^* \leqq N_P(Q)$, $N_P(Q) \not\leqq \mathcal{P}_P(L_i)$. From Lemma 11.5 $QL_i = L_iQ$ and hence, employing (2.6) upon QL_i, we have $L_i = YO_{\pi_i}(L_iQ)$. Since $q \neq 2$, $[[Q, \alpha_i], O_{\pi_i}(L_iQ)] = 1$ by (2.11'). From Lemma 7.9(f) $Q \neq Q_{\alpha_i}$ and therefore, using (2.3)(viii), we obtain

$$(N_G([Q, \alpha_i]))_{\{p \cup \pi_i\}} \geqq P_{\alpha_i}, Q_{\pi_i}(L_iQ).$$

This is the required contradiction as $P_{\alpha_i} \not\leqq \mathcal{P}_P(L_i)$ and $O_{\pi_i}(L_iQ) \not\leqq Y$.

(ii) This may be established by a similar argument.

LEMMA 11.11. *Suppose* $PL_i \neq L_iP$ *where* $i = 2$ *or* 3 *and* $Z(J(P)) \leqq N_P(Q)$. *Then either*

(i) $Z(J(P)) \leqq N_P(L_i)$; *or*

(ii) $q = 2$ *and* $Z(Q) \leqq Q_{\alpha_i}$.

PROOF. Without loss of generality we set $i = 3$. Suppose $Z(J(P)) \not\leqq N_P(L_3)$ and $q \neq 2$. Hence, by Lemma 11.10, $\mathfrak{M}(p, \pi_3) = \{P, N_P(L_3)L_3\}$ and so $P^*_{\langle p\sigma\rangle} \leqq N_P(L_3)$.

If (say) $Z(J(P))_\sigma \neq (Z(J(P))_\sigma)^*_{\langle \rho\tau \rangle}$, then, by (2.9) and (2.3)(i),

$$1 \neq [[Z(J(P))_\sigma, \rho], \tau] \leqq O_p(P_\sigma L_{3_\sigma}) \cap Z(J(P)).$$

Since $L_{3_\sigma} \trianglelefteq P_\sigma L_{3_\sigma}$ we then have

$$(N_G(O_p(P_\sigma L_{3_\sigma}) \cap Z(J(P))))_{\{p \cup \pi_3\}} \geqq Z(J(P)), L_{3_\sigma}.$$

The shape of $\mathfrak{M}(p, \pi_3)$ then forces $Z(J(P)) \leqq N_P(L_3)$, contrary to the present supposition. Therefore $Z(J(P))_\sigma = (Z(J(P))_\sigma)^*_{\langle \rho\tau \rangle}$. Similar argument also yields $Z(J(P))_\rho = (Z(J(P))_\rho)^*_{\langle \sigma\tau \rangle}$.

By Lemma 3.13(iii) $[P_{\rho\sigma}, L_3] = 1$ and so $(Z(J(P)))_{\rho\sigma} = 1$. Consequently $Z(J(P))_\sigma = (Z(J(P)))_{\sigma\tau}$ and $Z(J(P))_\rho = (Z(J(P)))_{\rho\tau}$. Hence $(Z(J(P)))^* = Z(J(P))_\tau$. Next we demonstrate that $Z(J(P)) \leqq P_\tau$. By (2.3)(i), $([Z(J(P)), \tau])^* = 1$. Since $Z(J(P)) \leqq N_P(Q)$, (2.9) yields $[Z(J(P)), \tau] \leqq O_p(N_P(Q)Q) = C_P(Q)$. Now $[L_3, Q_\sigma] = 1$ and consideration of $(N_G(Q_\sigma))_{\{p \cup \pi_3\}}$ implies that $[Z(J(P)), \tau] \leqq N_P(L_3)$. A further application of (2.9), this time to $L_3 N_P(L_3)$, gives

$$[Z(J(P)), \tau] \leqq O_p(N_P(L_3)L_3) = C_P(L_3).$$

Therefore, if $[Z(J(P)), \tau] \neq 1$ it would then follow that $Z(J(P)) \leqq N_P(L_3)$. Hence $Z(J(P)) \leqq P_\tau$.

Since $Z(J(P)) \leqq N_P(Q)$, clearly $N_P(Q) \not\leqq N_P(L_3)$. Hence, using (2.6) ($L_3$ and Q permute by Lemma 11.5) and the shape of $\mathfrak{M}(p, \pi_3)$, $L_3 \trianglelefteq L_3 Q$. By (2.11), $[L_3, [Q, \tau]] = 1$. From Lemma 7.9(f) $[Q, \tau] \neq 1$ and so (using (2.3)(viii))

$$(N_G([Q, \tau]))_{\{p \cup \pi_3\}} \geqq L_3, P_\tau,$$

whence $Z(J(P)) \leqq P_\tau \leqq N_P(L_3)$ whereas $Z(J(P)) \not\leqq N_P(L_3)$. Thus $Z(J(P)) \not\leqq N_P(L_3)$ implies that $q = 2$.

Arguing as in the previous paragraph, and using (2.12) instead of (2.11), it may be shown that $Z(J(P)) \not\leqq N_P(L_3)$ also implies $Z(Q) \leqq Q_\tau$. This proves the lemma.

Proof of Theorem 11.2. Suppose $(G, \langle \alpha \rangle)$ satisfies Hypotheses III and 11.1. By Lemma 11.9 $G = HK$. First we show that

(11.5) $$Z(P) \leqq N_P(K).$$

For any (non-trivial) α-invariant Sylow subgroup Q of K_0, $Z(P)(= Z(P)_{\sigma\tau}) \leqq N_P(Q)$. Thus $Z(P) \leqq N_P(K_0)$. Suppose we have $PL_2 \neq L_2 P$. Now $Z(P) \leqq P_{\sigma\tau}$ rules out the possibility $L_2^* \leqq N_{L_2}(P)$ and so, by Lemmas 7.6(iv) and 7.7(g), $Z(P) \leqq N_P(L_2)$. Similar considerations apply if $PL_3 \neq L_3 P$. If $PL_{23} \neq L_{23} P$, then (as $C_P(L_{23})\neq 1$) $Z(P)\leqq N_P(L_{23})$ by Lemmas 7.1 and 11.10. Thus, whatever the form of K, $Z(P)\leqq N_P(K)$.

Next we claim that

$$(11.6)\qquad \begin{cases} Z(J(P))_\rho = (Z(J(P))_\rho)^*_{\langle\sigma\tau\rangle}, Z(J(P))_\sigma = (Z(J(P))_\sigma)^*_{\langle\rho\tau\rangle}, \\ Z(J(P))_\tau = (Z(J(P))_\tau)^*_{\langle\rho\sigma\rangle} \text{ and } O_p(H)\neq 1 \text{ cannot hold.} \end{cases}$$

Since $O_P(H)\neq 1$, $R = Z(P)\cap O_p(H)\neq 1$. Applying Lemma 6.3 to $Z(J(P))(N_H(J(P)))_{p'}$ and using (2.6) on H gives $R\leqq Z(H)$. Then $1\neq R^G = R^{HK} = R^K\leqq N_G(K)$ by (11.1). This verifies (11.6).

Now suppose $p\neq 2$. Since P is of n-p type I, P is not star-covered and thus $O_p(H)\neq 1$ by Theorem 4.4. By (11.6) we may suppose that (say) $Z(J(P))_\rho\neq(Z(J(P))_\rho)^*_{\langle\sigma\tau\rangle}$. Consequently, by (2.3)(i) and (2.9), $Z(J(P))\cap O_p(P_\rho Q_\rho)\neq 1$ where Q is any non-trivial α-invariant Sylow q-subgroup of K_0. Hence the shape of $\mathfrak{M}(p,q)$ forces $Z(J(P))\leqq N_P(Q)$, and so $Z(J(P))\leqq N_P(K_0)$. We claim that $Z(J(P))\leqq N_P(K)$. Since $p\neq 2$, $H_0 = P$ by Lemma 10.4. If $H = PL_1$, then, as $[P_{\sigma\tau}, L_1] = 1$, $Z(P)\leqq Z(H)$ which, by (11.5), implies that G does not satisfy Hypothesis III. So we may suppose that $PL_2 = L_2P$ and $L_2\neq 1$.

Observe, if P permutes with both L_2 and L_3, then $K = K_0L_{23}$ and hence, using Lemma 7.1 and the fact that $[K_0, L_{23}] = 1$, we have $Z(J(P))\leqq N_P(K)$. Thus, in order to show that $Z(J(P))\leqq N_P(K)$, we must deal with the situation $PL_2 = L_2P$, $L_2\neq 1$ and $PL_3\neq L_3P$ and deduce that $Z(J(P))\leqq N_P(L_3)$. Let Q be a non-trivial α-invariant Sylow subgroup of K_0. If $Z(J(P))\not\leqq N_P(L_3)$, then Lemma 11.11 asserts that $Z(Q)\leqq Q_\tau$ which, as $[Q_\tau, L_2] = 1$, implies that $QL_2 = L_2Q$. But then $L_2 = 1$ by Lemma 10.6(v), a contradiction. Therefore $Z(J(P))\leqq N_P(K)$.

Set $S = (Z(P)\cap O_P(H))^H$. Note that $S\neq 1$ and that, by (2.6), $S\leqq Z(J(P))$. Consequently $S^G = S^{HK} = S^K\leqq N_G(K)$, and so $(G,\langle\alpha\rangle)$ cannot satisfy Hypothesis III.

Thus we may assume that the factorization in Lemma 11.9 cannot be achieved when $p\neq 2$. Hence, the set of α-invariant Sylow q-subgroups of G of type Ψ, with q odd, must generate a soluble Hall subgroup of G, namely $K_0(H_0)_{2'}$. Also, because of Hypothesis 11.1, we have $2\in\pi(H_0)$.

(11.7) $K(H_0)_{2'} = (H_0)_{2'}K$ is a soluble Hall subgroup of G.

Let W and Q be, respectively, α-invariant Sylow w- and q-subgroups of $(H_0)_{2'}$ and K_0. So W permutes with both T and Q and Q is of n-p type II with respect to T. Suppose $L_3T\neq TL_3$ (so $L_3\leqq K$). We show that $WL_3 = L_3W$. Assuming $WL_3\neq L_3W$ we deduce a contradiction. If $L_3^*\leqq N_{L_3}(W)$ pertains, then Lemma 5.8(e)(ii) and the shape of $\mathfrak{M}(w,\pi_3)$ contradicts Lemma 3.14. So

$W^*_{\langle\rho\sigma\rangle} \leqq N_W(L_3)$. Since $Q_\sigma \not\leqq Q_\rho$ and $[Q_\sigma, L_3] = 1$, using Lemma 5.8(b) we obtain $O_w(WQ) \leqq N_W(L_3)$. By Lemmas 4.6 and 7.6(i)(d) this gives $W = W_\tau$. Employing (2.3)(ix) and Lemma 10.3 gives the impossible $T = O_2(TW)T_\tau \leqq N_T(Q)$. Hence $WL_3 = L_3W$. Similar arguments apply if $L_2T \neq TL_2$.

Suppose $L_{23}T \neq TL_{23}$. If $WL_{23} \neq L_{23}W$, then $W_\rho \leqq C_W(L_{23})$. Since $[Q, L_{23}] = 1$ and $Q \neq Q_\rho$, similar reasoning (and use of (2.3)(v)) gives $W = W_\sigma$ or W_τ from which we obtain the contradiction as before.

Thus we have established (11.7).

In view of $(G, \langle\alpha\rangle)$ satisfying Hypothesis III, $O_{\pi(K)'}(K(H_0)_{2'}) = 1$. Hence, employing [Theorem 1; 1], we have $Z(J(K)) \trianglelefteq K(H_0)_{2'}$. Since $Z(J(K))$ ch K it follows from (11.1) that $Z(T) \leqq N_G(Z(J(K)))$. Because $Z(T) \leqq Z(TL_1)$, $H \neq H_0L_1$. So we may suppose $1 \neq L_2 \leqq H$. Lemma 10.6(iv) and (v) implies that $H = H_0L_1L_2$. Set $\tilde{H} = TL_1L_2$. Since $O_{\pi_2}(TL_2) = 1$, we have, using (2.6), $\tilde{H} = N_{\tilde{H}}(J(T))C_{\tilde{H}}(Z(T))$. Therefore $U = Z(T)^{\tilde{H}} \leqq Z(J(T))$, and hence $O_2(H) \neq 1$. Thus we may use (11.6) for the factorization $G = H(K(H_0)_{2'})$ (since $Z(T), K(H_0)_{2'} \leqq N_G(Z(J(K))))$ to deduce that $Z(J(T)) \leqq N_T(K_0)$. Then, with the compliance of Lemma 11.11, we may obtain $Z(J(T)) \leqq N_T(K)$. This yields the contradiction

$$1 \neq U^G = U^{\tilde{H}(K(H_0)_{2'})} = U^{K(H_0)_{2'}} \leqq N_G(Z(J(K))) \neq G.$$

This completes the proof of Theorem 11.2.

12. α-Invariant Sylow subgroups of n-p type III and IV

First we state an appropriate hypothesis.

HYPOTHESIS 12.1. G possesses α-invariant Sylow subgroups of n-p type III and IV but none of n-p type I and II.

The object of this section is the proof of

THEOREM 12.2. *Hypothesis* 12.1 *is incompatible with Hypothesis III.*

In section 10 certain elementary deductions were made concerning n-p type "III–IV interactions". Before tackling the proof of Theorem 12.2 we continue in a similar, though more involved, vein. From now until the end of Lemma 12.9 we assume Hypothesis 12.1 to hold.

Throughout this section Q will denote an α-invariant Sylow q-subgroup of G of type Ψ which is of n-p type IV with respect to T. We also fix the following notation: $T_\rho \leqq \mathcal{P}_T(Q) = X$ and $Q^*_{\langle\sigma\tau\rangle} \leqq \mathcal{P}_Q(T) = Y$. We note that, if U is any

α-invariant Sylow subgroup of type Ψ and n-p type IV (with respect to T), then $T_\rho \leqq \mathcal{P}_T(U)$ and $U^*_{\langle\sigma\tau\rangle} \leqq \mathcal{P}_U(T)$.

LEMMA 12.3. *Suppose W is an α-invariant Sylow w-subgroup of G of type Ψ with $w \neq 2$. If $WT = TW$, then $L_1W = WL_1$.*

PROOF. We first recall, from Lemma 10.8(i), that L_1 and T permute. Further, by Hypothesis 12.1 and Lemma 7.8, Q and W permute.

Supposing that $L_1W \neq WL_1$ pertains we shall derive a contradiction. Now Corollary 7.4 predicts (since $2 \notin \{q, w\}$) that either $L_1^* \leqq \mathcal{P}_{L_1}(W)$ or $W^*_{\langle\sigma\tau\rangle} \leqq \mathcal{P}_W(L_1)$.

First we consider the alternative $L_1^* \leqq \mathcal{P}_{L_1}(W)$. So $\mathfrak{M}(\pi_1, w) = \{N_{L_1}(W)W, L_1\}$. Employing Lemma 5.8(e)(i) (since T permutes with both L_1 and W, $2 \notin \pi_1$ and $W(TL_1) \neq G$) yields that $L_1 = \mathcal{P}_{L_1}(W)C_{L_1}(T) = N_{L_1}(W)C_{L_1}(T)$. Because of the shape of $\mathfrak{M}(\pi_1, w)$ we have $N_W(J(T)) = C_W(Z(T)) = 1$. Consequently $W \trianglelefteq WT$ by (2.6). From Lemma 7.5 $W_{\sigma\tau} = 1$ and hence, since $Y \geqq Q^*_{\langle\sigma\tau\rangle}$ and $T(QW) \neq G$, another application of Lemma 5.8(a)(i) gives $Q = YC_Q(W)$. But then $W \trianglelefteq WT$ implies that $Q = Y$, against $QT \neq TQ$. Therefore $L_1^* \not\leqq \mathcal{P}_{L_1}(W)$.

So $W^*_{\langle\sigma\tau\rangle} \leqq \mathcal{P}_W(L_1) = N_W(L_1)$. If $[O_w(TW), \sigma] \neq 1$, then (since $Q_\sigma \leqq Y$) Lemma 5.8(c) dictates that one of $O_q(QW) \leqq Y$ and $T_\sigma \leqq X$, must hold. The former possibility yields that $Q = Q^*O_q(QW) = Q^*Y = Q_\rho$ (since $Y \leqq Q_\rho = Q^*$) whereas $Q \neq Q_\rho$. Whilst $T_\sigma \leqq X$ forces $TQ = QT$ by Lemma 7.8. Therefore $O_w(TW) \leqq W_\sigma$ and, similarly, $O_w(TW) \leqq W_\tau$. So $[T, O_w(TW)] = 1$ by (2.3)(xi). Also, by Theorem 4.4 and Lemma 3.3(vii), W is star-covered. Hence, from Lemma 7.6, $N_W(L_1) \leqq W_\rho = W^*$, and so $W = W_\rho$. Clearly then we have $W_{\sigma\tau} = 1$, $O_w(WT) = 1$ and $\mathfrak{M}(\pi_1, w) = \{W, N_W(L_1)L_1\}$.

We claim that $L_1Q \neq QL_1$. For suppose $L_1Q = QL_1$ were to hold, then applying Lemma 5.8(e)(i) to L_1, Q and W (since $W^*_{\langle\sigma\tau\rangle} \leqq N_W(L_1)$, $Q_{\sigma\tau} = 1$ and $L_1(QW) \neq G$) yields $W = N_W(L_1)C_W(Q)$. However $Q \neq Q_\rho = Q^*$ means $O_q(QL_1) \neq 1$ by Corollary 4.5 which in turn forces $C_W(Q) \leqq N_W(L_1)$, contrary to $WL_1 \neq L_1W$. Hence the claim is verified.

Moreover $L_1^* \leqq \mathcal{P}_{L_1}(Q)$ is untenable. For $L_1^* \leqq \mathcal{P}_{L_1}(Q)$ implies by (2.3)(xi), as $\mathcal{P}_{L_1}(Q) = N_{L_1}(Q)$, $L_{1_\rho} = 1$ and $Q^* = Q_\rho$, that $[L_{1_\sigma}, Q_\sigma] = [L_{1_\tau}, Q_\tau] = 1$. If $L_{1_\sigma} = L_{1_\tau}$, then, using Lemma 3.6, $Q^*_{\langle\sigma\tau\rangle} \leqq C_Q(L_1^*) = C_Q(L_1)$ whereas $\mathcal{P}_Q(L_1) = 1$. On the other hand, $L_{1_\sigma} \neq L_{1_\tau}$ gives, by Lemma 7.5(d) and (g), that $Z(L_1) \leqq L_{1_{\sigma\tau}}$ which again yields the impossible $Q^*_{\langle\sigma\tau\rangle} \leqq \mathcal{P}_Q(L_1)$. Thus $L_1^* \not\leqq \mathcal{P}_{L_1}(Q)$, and so $Q^*_{\langle\sigma\tau\rangle} \leqq \mathcal{P}_Q(L_1) = N_Q(L_1)$.

We recall that $[L_1, Z(T)] = 1$ and $O_w(WT) = 1$, whence, by (2.6), $W = N_W(J(T))C_W(Z(T)) = N_W(J(T))N_W(L_1)$. Because $Z(T) \leqq T_{\sigma\tau}$ and $Z(T) \leqq$

$Z(J(T))$, we have $[Z(J(T)),\rho]\neq 1$. Now $W=W_\rho$ and so, applying (2.3)(ix) to $Z(J(T))N_W(J(T))$ we have that $[Z(J(T)),\rho]\trianglelefteq Z(J(T))N_W(J(T))$. With the aid of (2.12) it follows that $[O_{\pi_1}(L_1T),[Z(J(T)),\rho]]=1$ from which we infer that either $O_{\pi_1}(L_1T)\leqq \mathcal{P}_{L_1}(W)$ or $W=N_W(L_1)$. Evidently the former holds and so, as $\mathcal{P}_{L_1}(W)=1$, we may assert that L_1 is star-covered.

Now $N_Q(L_1)$ normalizes $O_w(WQ)\cap N_W(L_1)\geqq (O_w(WQ))^*_{\langle\sigma\tau\rangle}$ and so, since $Q_{\sigma\tau}=1$,

$$W=N_W(L_1)C_W(N_Q(L_1)).$$

Set $\bar{L}_1=L_1/\Phi(L_1)$. Then $\bar{L}_1=\bar{L}_{1_\sigma}\bar{L}_{1_\tau}$ as L_1 is star-covered. Since $Q_{\sigma\tau}=1$ and $Q^*_{\langle\sigma\tau\rangle}\leqq N_Q(L_1)$, $\bar{L}_1=(\bar{L}_1)_\sigma C_{\bar{L}_1}(Q_\sigma)=(\bar{L}_1)_\tau C_{\bar{L}_1}(Q_\tau)$. Now

$$C_{L_1}(Q_\sigma),\ C_W(N_Q(L_2))\leqq (C_G(Q_\sigma))_{\{\pi_1,w\}}$$

implies, as $W=N_W(L_1)C_W(N_Q(L_2))$, $C_{L_1}(Q_\sigma)\leqq \mathcal{P}_{L_1}(W)=1$. Hence $\bar{L}_1=(\bar{L}_1)_\sigma$ and, similarly, $\bar{L}_1=(\bar{L}_1)_\tau$. Therefore $L_1=L_{1_{\sigma\tau}}$ by [Theorem 5.1.4; 2]. This, by (2.3)(xi) and (2.21)(v), contradicts the deduction $L_1Q\neq QL_1$.

With this contradiction we have established Lemma 12.3.

LEMMA 12.4. *Suppose $TL_i=L_iT$ where $i=2$ or 3. Then $L_1L_i=L_iL_1$.*

PROOF. We shall show that the assumption $L_1L_i\neq L_iL_1$ leads to a contradiction. Without loss of generality we may take $i=2$. We recall that $[Z(T),L_1]=1$, that $TL_1=L_1T$ and that $Y\leqq Q_\rho=Q^*\neq Q$.

Now, by Theorem 8.1, we see that at least one of $L_1Q=QL_1$ and $L_2Q=QL_2$ must occur. We shall examine these possibilities in a moment. But first we state two observations.

(12.1) Suppose $L_1Q=QL_1$ holds. Then
(i) $O_{\pi_1}(L_1T)\leqq L_{1_{\sigma\tau}}$;
(ii) $[O_{\pi_1}(L_1T),T]=1$; and
(iii) L_1 is star-covered.

If (say) $[O_{\pi_1}(L_1T),\sigma]\neq 1$ then, appealing to Lemma 5.8(c), we have one of $T_\sigma\leqq X$ and $O_q(QL_1)\leqq Y$ holding. The former forces $TQ=QT$ (see Lemma 7.8) and the latter gives $Q=O_q(QL_1)Q^*=YQ^*=Q_\rho$ which is not possible. So $O_{\pi_1}(L_1T)\leqq L_{1_\sigma}$ and likewise $O_{\pi_1}(L_1T)\leqq L_{1_\tau}$, so giving (i). Now part (ii) follows from (i) by (2.3)(xi) and part (iii) from (i), Theorem 4.4 and Lemma 3.3(vii).

(12.2) Suppose $L_2Q=QL_2$ holds. Then
(i) $Q=YC_Q(L_2)$;

(ii) $O_{\pi_2}(L_2T)=1$ and so L_2 is star-covered; and

(iii) $L_2=C_{L_2}(Z(T))N_{L_2}(J(T))$.

Part (i), since $Y \geqq Q_\sigma$, $L_{2_\sigma}=1$, and $TL_2=L_2T$ is just a consequence of Lemma 5.8(e). While (ii) follows immediately from (i) and (2.6) yields (iii).

Case 1. $QL_1=L_1Q$ and $QL_2=L_2Q$

Since, by assumption, $L_1L_2 \neq L_2L_1$ we have at least one of $L_{1_\tau} \leqq N_{L_1}(L_2)$ and $L_{2_\tau} \leqq N_{L_2}(L_1)$ with $\mathfrak{M}(\pi_1,\pi_2)=\{L_1N_{L_2}(L_1), L_2N_{L_1}(L_2)\}$. First we consider the possibility $L_{2_\tau} \leqq N_{L_2}(L_1)$. Applying Lemma 5.8(e) to L_1, L_2 and Q (since $(L_2)^*_{\langle\sigma\tau\rangle} \leqq N_{L_2}(L_1)=\mathcal{P}_{L_2}(L_1)$ and $Q_{\sigma\tau}=1$) gives $L_2=N_{L_2}(L_1)C_{L_2}(Q)$. So forcing $O_q(QL_1)=1$ to hold. Hence $Q=Q_\rho$ by (2.13), a contradiction.

Now we consider $L_{1_\tau} \leqq N_{L_1}(L_2)$. Applying Lemma 5.8(b) to the triple Q, L_1 and T we obtain (as $L_{1_\rho}=1$ and $X \geqq T_\rho$)

$$O_2(TL_1)=C_{O_2(TL_1)}(L_1)(O_2(TL_1)\cap X).$$

Employing Lemma 5.8(e) on the same triple yields (since $Q^*_{\langle\sigma\tau\rangle} \leqq Y$) that

$$O_q(QL_1)=C_{O_q(QL_1)}([L_1,\sigma\tau])(O_q(QL_1)\cap Y).$$

Because $Q=Q_\rho O_q(QL_1)$, $[L_1,\sigma\tau]\neq 1$ and $Q \neq Q_\rho$, we conclude that $C_Q([L_1,\sigma\tau]) \nleqq Y$, and therefore $O_2(TL_1) \leqq X$ holds.

From (12.1) we have that L_1 is star-covered and so $N_{L_1}(L_2) \leqq L_{1_\sigma}$ (else L_1 would not be star-covered). Thus $L_1=L_1^*=L_{1_\sigma}$. Consequently, using (2.3)(ix), $T=T_\sigma O_2(TL_1)=T_\sigma X$. This contradicts Lemma 7.10(g) and thus we have ruled out case 1.

Case 2. $QL_2=L_2Q$ and $QL_1 \neq L_1Q$

First we examine the possibility $L_1^* \leqq N_{L_1}(Q)$. Since, by Lemma 7.10, $Z(T) \leqq N_T(Q)$, we have $Z(T)$ normalizing $O_{\pi_1}(L_1T)\cap N_{L_1}(Q)$. So $O_{\pi_1}(L_1T)=C_{O_{\pi_1}(L_1T)}(Z(T))(O_{\pi_1}(L_1T)\cap N_{L_1}(Q))$ by (2.14)(ii). Thus $L_1=L_1^*O_{\pi_1}(L_1T)=N_{L_1}(Q)C_{L_1}(Z(T))$. From Lemma 7.10(f) $[Z(T),Y]=1$ and so the shape of $\mathfrak{M}(\pi_1,q)$ forces $L_1=N_{L_1}(Q)$.

Therefore we may assume $Q^*_{\langle\sigma\tau\rangle} \leqq N_Q(L_1)$ holds. Observe that $Q=YN_Q(L_1)$ is impossible. For then $Q=YN_Q(L_1)=Q_\rho N_Q(L_1)$ whence, by Lemma 7.6 and (2.3)(ix),

$$[Q,\rho]=[N_Q(L_1),\rho] \leqq C_Q(L_1).$$

So $(N_G([Q,\rho]))_{\{q,\pi_1\}} \geqq Q, L_1$, which is not the case. In particular, this means that $L_{1_\tau} \leqq N_{L_1}(L_2)$ is untenable because, since $L_{1_\tau} \nleqq N_{L_1}(Q)$, $L_{1_\tau} \leqq N_{L_1}(L_2)$ would imply $C_Q(L_2) \leqq N_Q(L_1)$ which then implies $Q=YN_Q(L_1)$ by (12.2)(i).

Therefore $L_{2_\tau} \leqq N_{L_2}(L_1)$ holds. By (12.2) L_2 is star-covered and so $N_{L_2}(L_1) \leqq L_{2_\rho} = L_2^* = L_2$. Hence $N_{L_1}(L_2) = 1$ as $L_{1_\rho} = 1$. Now $N_{L_2Q}(L_1) \geqq (L_2Q)^*_{\langle\sigma\tau\rangle}$ and so (using Corollary 5.4 and (2.14)(ii)) $L_2 = N_{L_2}(L_1)C_{L_2}(N_Q(L_1))$. Clearly we must have $C_Q(L_1) = 1$.

Because $L_2 = L_{2_\rho}$, $T = T_\rho O_2(TL_2)$. Thus $O_2(TL_2) \nleqq X$. So, by Lemma 7.10(c), $(Z(O_2(TL_2)))_\rho = 1$. Therefore $Z(O_2(TL_2))O_{\pi_1}(TL_1)$ admits ρ fixed-point-freely. So $[Z(O_2(TL_2)), O_{\pi_1}(TL_1)] = 1$ and thus $O_{\pi_1}(TL_1) \leqq N_{L_1}(L_2) = 1$. Therefore L_1 is star-covered by Theorem 4.4. Recalling that $Q^*_{\langle\sigma\tau\rangle} \leqq N_Q(L_1)$ and $C_Q(L_1) = 1$ we see, using Lemma 7.6(iii)(c), that $Q = Q_\rho$. With this contradiction we have disposed of case 2.

Case 3. $QL_1 = L_1Q$ and $QL_2 \neq L_2Q$

Combining the facts $Q^*_{\langle\sigma\tau\rangle} \leqq Y \leqq Q_\rho$ and $Q = Q_\rho O_q(QL_1)$ with Lemma 5.8(b) (applied to L_1, Q and T) we obtain $Q = Q_\rho C_Q([L_1, \sigma\tau])$. A further application of Lemma 5.8(b), since $L_{1_\rho} = 1$ and $T_\rho \leqq X$, yields

$$O_2(TL_1) = C_{O_2(TL_1)}(L_1)(O_2(TL_1) \cap X).$$

Since $[L_1, \sigma\tau] \neq 1$ (otherwise $L_1L_2 = L_2L_1$) and $Q \neq Q_\rho$ we may infer that $O_2(TL_1) \leqq X$.

The possibility that $L_2^* \leqq N_{L_2}(Q)$ may be eliminated as in case 2. So $Q^*_{\langle\rho\tau\rangle} \leqq N_Q(L_2)$.

We assert that $Q \neq Q_\rho N_Q(L_2)$. For $Q = Q_\rho N_Q(L_2)$ when combined with (see (2.13)) $N_Q(L_2) = C_Q(L_2)(N_Q(L_2))_\sigma$ gives $Q = Q_\rho Q_\sigma C_Q(L_2) = Q_\rho C_Q(L_2)$, and so $[Q, \rho] \leqq C_Q(L_2)$. As in case 2 this leads to the untenable $Q = Q_\rho$.

Suppose, for the moment, that $L_{2_\tau} \leqq N_{L_2}(L_1)$. Then, using (2.3)(viii)

$$L_{2_\tau}, C_Q([L_1, \tau]) \leqq (N_G([L_1, \tau]))_{\{\pi_2, q\}}.$$

Thus either $[L_1, \tau] = 1$ or $C_Q([L_1, \tau]) \leqq N_Q(L_2)$. The latter would give $Q = Q_\rho C_Q([L_1, \sigma\tau]) = Q_\rho N_Q(L_2)$. So $L_1 = L_{1_\tau}$. Consequently, by (2.3)(ix), $T = T_\tau O_2(TL_1) = T_\tau X$, which is contrary to Lemma 7.10(g).

While $L_{1_\tau} \leqq N_{L_1}(L_2)$ implies, as L_1 is star-covered by (12.1), that $L_1 = L_1^* = L_{1_\sigma}$. This then gives $T = O_2(TL_1)T_\sigma = XT_\sigma$, again contradicting Lemma 7.10(g). This completes case 3 and also finishes the proof of Lemma 12.4.

We introduce the following notation:

$\tilde{L}_0 = \langle P \mid P$ is an α-invariant Sylow p-subgroup of G of type Ψ with $p \neq 2\rangle$

LEMMA 12.5. (i) *$\tilde{L}_0$ is an α-invariant soluble Hall subgroup of G of odd order.*
(ii) *$L_{23}\tilde{L}_0 = \tilde{L}_0L_{23}$ is an α-invariant soluble Hall subgroup of G.*

PROOF. (i) This follows from Hypothesis 12.1 and (2.4).

(ii) Let W be an α-invariant Sylow subgroup of $\tilde{L}_0$. By (2.4) and (2.5) it suffices to show that $WL_{23} = L_{23}W$. From Lemma 10.8(ii) Q permutes with L_{23}. So we may suppose $W \neq Q$. As $Q \neq Q_p = Q^*$ we have $O_q(QW) \neq 1$ by (i) and Corollary 4.5. Since, by (2.8), $[Q, L_{23}] = 1$ this yields that $WL_{23} = L_{23}W$.

LEMMA 12.6. *Suppose $TL_i = L_iT$ where $i = 2$ or 3. Then $L_i = L_{i_p}$.*

PROOF. We break the proof into two parts depending on whether $L_iQ \neq QL_i$ or $L_iQ = QL_i$. Without loss of generality we take $i = 2$.

Case 1. $L_2Q \neq QL_2$

So, by Corollary 7.4, one of $L_2^* \leqq \mathcal{P}_{L_2}(Q)$ and $Q^*_{\langle \sigma\tau\rangle} \leqq \mathcal{P}_Q(L_2)$ must hold. If the former pertains, then $\mathfrak{M}(\pi_2, q) = \{L_2, N_{L_2}(Q)Q\}$. Since, by Lemma 7.10, $Z(T) \leqq N_T(Q)$ we have $Z(T)$ normalizing

$$O_{\pi_2}(L_2T) \cap N_{L_2}(Q) \geqq (O_{\pi_2}(L_2T))^*.$$

Thus $L_2 = C_{L_2}(Z(T))N_{L_2}(Q)$ by (2.14)(ii) and Corollary 4.5. Recall from Lemma 7.10 that $[Z(T), \mathcal{P}_Q(T)] = 1$ which, when combined with the shape of $\mathfrak{M}(\pi_2, q)$ and the fact that $\mathcal{P}_Q(T) \neq 1$, forces $L_2 = N_{L_2}(Q)$. Hence we conclude that $Q^*_{\langle \rho\tau\rangle} \leqq \mathcal{P}_Q(L_2)$ holds. Thus $Q^* = Q_p \leqq \mathcal{P}_Q(L_2)$, and so $\mathfrak{M}(\pi_2, q) = \{L_2N_Q(L_2), Q\}$ by Lemma 5.1(d) and (2.21)(vi).

Now, since $Q^* \nleqq Y$ we infer, with the aid of (2.7), that $T = O_2(TL_2)X$. Thus, in view of Lemma 7.10(c), we have $(Z(O_2(TL_2)))_p = 1$. Hence, by (2.11), $[[L_2, \rho], Z(O_2(TL_2))] = 1$. Suppose $[L_2, \rho] \neq 1$. Then we have, using (2.3)(viii),

$$(N_G([L_2, \rho]))_{\{2,p\}} \geqq Z(O_2(TL_2)), Q_p.$$

Hence $Z(O_2(TL_2)) \leqq X$ and, furthermore, $Z(O_2(TL_2))$ normalizes $O_q(QX) \cap N_Q([L_2, \rho])(\geqq O_q(QX)^*)$. Therefore

$$O_q(QX) = C_{O_q(QX)}(Z(O_2(TL_2)))(O_q(QX) \cap N_Q([L_2, \rho])),$$

and so

$$\begin{aligned} Q = Q_pO_q(QX) &= Q_pC_Q(Z(O_2(TL_2)))N_Q([L_2, \rho]) \\ &= Q_pYN_Q([L_2, \rho]) \\ &= N_Q([L_2, \rho]). \end{aligned}$$

But this clearly contradicts the supposition $QL_2 \neq L_2Q$. Thus, when $QL_2 \neq L_2Q$, we have shown that $L_2 = L_{2_p}$.

Case 2. $L_2Q = QL_2$

Clearly we may suppose that $L_2 \neq 1$. First we note that $(L_2T)Q \neq G$ (and hence $T(L_2Q) \neq G$ also). For suppose $(L_2T)Q = G$. Then $O_{\pi_2}(L_2Q) = 1$, for otherwise $(O_{\pi_2}(LQ))^G$ would be a non-trivial proper α-invariant normal subgroup of G. Thus $Q \trianglelefteq QL_2$ by (2.10)(i). Since $Z(T) \leqq N_T(Q)$ we then have $(Z(T))^G = (Z(T))^{QL_2} \leqq N_G(Q) \neq G$, contrary to $(G, \langle\alpha\rangle)$ satisfying Hypothesis III.

Now a double application of Lemma 5.8(a) to the triple T, L_2 and Q gives

$$O_2(TL_2) = C_{O_2(TL_2)}([L_2, \rho])(O_2(TL_2) \cap X), \qquad \text{and}$$

$$O_q(QL_2) = C_{O_q(QL_2)}(L_2)(O_q(QL_2) \cap Y').$$

Consequently $Q = Q^*O_q(QL_2) = Q_\rho C_Q(L_2)$. Since $Q \neq Q_\rho$ and $Y \leqq Q_\rho$, we may assert that $T = O_2(TL_2)X$ by (2.7). Therefore we have $T = C_T([L_2, \rho])X$. Because $Q \neq Q_\rho$ and $TQ \neq QT$ the only possible conclusion is $[L_2, \rho] = 1$. This completes case 2, and the proof of the lemma.

The next result is an immediate consequence of Lemma 12.6.

LEMMA 12.7. *Let T be of n-p type III with respect to Q. If both L_2 and L_3 permute with T, then $L_2L_3 = L_3L_2$.*

PROOF. From Lemma 12.6 we have $L_2, L_3 \leqq G_\rho$, and so, since G_ρ is soluble, the lemma follows.

LEMMA 12.8. *Let U be an α-invariant Sylow u-subgroup of G of type Ψ and n-p type IV. If $TL_i = L_iT$ and $QL_i \neq L_iQ$ where $i = 2$ or 3, then $UL_i \neq L_iU$.*

PROOF. Suppose the lemma is false. So $L_iU = UL_i$. By Lemma 12.5(i), as $u \neq 2$, we have $UQ = QU$. Since $TL_i = L_iT$, from Lemma 12.6, $L_i = L_{i\rho}$. By (2.3)(ix) this gives $[U, \rho] \trianglelefteq L_iU$. Moreover, since $U^* = U_\rho \neq U$ by Lemma 7.10, Corollary 4.5 implies that $1 \neq [U, \rho] \leqq O_u(UQ)$. Considering $(N_G([U, \rho]))_{\{\pi_i, q\}}$ we obtain $O_q(QU) \leqq \mathcal{P}_Q(L_i)$. Because $L_i = L_{i_\rho}$ we further have $Q_\rho \leqq \mathcal{P}_Q(L_i)$, and so $Q = Q_\rho O_q(QU) \leqq \mathcal{P}_Q(L_1)$. With this contradiction the lemma is verified.

LEMMA 12.9. *Let W be an α-invariant Sylow w-subgroup of G of type Ψ with $w \neq 2$. If $WT = TW$, then $WL_2 = L_2W$ and $WL_3 = L_3W$.*

PROOF. Suppose that (say) $WL_2 \neq L_2W$ and argue for a contradiction. From Lemma 12.5(i) we have $WQ = QW$. We first establish that

(12.3) $\quad O_w(WQ) \leqq N_W(L_2)$ and $W^*_{\langle \rho\tau\rangle} \leqq N_W(L_2)$ cannot hold.

Suppose $O_w(WQ)$, $W^*_{\langle\rho\tau\rangle} \leqq N_W(L_2)$ did hold. Then, by Lemmas 4.6 and 7.6 and Corollary 4.5, $W = W_\sigma$. Hence, employing (2.3)(ix), we have $T = O_2(TW)T_\sigma$. Since $(TW)Q \neq G \neq T(WQ)$, Lemma 5.8(b) yields that

$$O_2(TW) = C_{O_2(TW)}([W, \rho])(X \cap O_2(TW)) \qquad \text{and}$$

$$O_q(QW) = C_{O_q(QW)}([W, \tau])(Y \cap O_q(QW)).$$

Thus $T = T_\sigma X C_T([W, \rho])$ and $Q = C_Q([W, \tau])Q_\rho$. By Lemma 7.10 $Q \neq Q_\rho$ and $T \neq T_\sigma X$ and so we deduce that $[W, \rho] \cap [W, \tau] = 1$. Therefore $W = W_\rho W_\tau$. Employing (2.10)(ii) and Lemma 6.1 yields that G possesses a normal w-complement which, since $W \neq 1$, is contrary to Hypothesis III holding. This verifies (12.3).

Now suppose that $L_2 Q = QL_2$. Since $Q \neq Q_\rho = Q^*$, $O_q(QW) \neq 1$ by Corollary 4.5. Hence $L_2^* \leqq N_{L_2}(W) = \mathcal{P}_{L_2}(W)$ is untenable by Lemma 5.8(f). Therefore $W^*_{\langle\rho\tau\rangle} \leqq N_W(L_2) = \mathcal{P}_W(L_2)$. Since $L_{2_\rho} \not\leqq \mathcal{P}_{L_2}(W)$ by Corollary 7.4 and $O_q(QL_2) \not\leqq Q_\rho$, Lemma 5.8(c) implies that $O_w(WQ) \leqq N_W(L_2)$. From (12.3) we infer that $L_2 Q \neq QL_2$. In view of Lemma 10.9 this gives $TL_2 = L_2 T$ and so, by Lemma 12.6, $L_2 = L_{2_\rho}$. Clearly we then have $W^*_{\langle\rho\tau\rangle} \leqq N_W(L_2)$ and $Q^*_{\langle\rho\tau\rangle} \leqq N_Q(L_2) = \mathcal{P}_Q(L_2)$. Also we note that $1 \neq Q_\tau \leqq [N_Q(L_2), \rho\sigma] \leqq C_Q(L_2)$. Using (2.14)(i) we deduce that

$$O_w(WQ) = C_{O_w(WQ)}([N_Q(L_2), \rho])(O_w(WQ) \cap N_W(L_2)).$$

Since $1 \neq [N_Q(L_2), \rho] \leqq C_Q(L_2)$ (otherwise $Q = Q_\rho$ by (2.3)(v)) this gives $O_w(WQ) \leqq N_W(L_2)$. By (12.3) we have a contradiction.

The proof of Lemma 12.9 is complete.

We are now in a position to prove Theorem 12.2.

Proof of Theorem 12.2. Assume Hypotheses III and 12.1 hold, and argue for a contradiction. We introduce the following notation:

$$\tilde{L}_0^+ = \langle W \mid W \text{ is } \alpha\text{-invariant Sylow subgroup of } \tilde{L}_0 \text{ with } TW = WT \rangle,$$

$$\tilde{L}_0^- = \langle W \mid W \text{ is } \alpha\text{-invariant Sylow subgroup of } \tilde{L}_0 \text{ with } TW \neq WT \rangle.$$

Also H will denote the subgroup of G generated by $\tilde{L}_0^+$ and those of $\{L_i, L_{jk} \mid i, j, k \in \Psi\}$ which permute with T, and K will denote the subgroup of G generated by $\tilde{L}_0^-$ and those of $\{L_i, L_{jk} \mid i, j, k \in \Psi\}$ which do not permute with T.

The combined effect of (2.4), (2.5), Theorem 8.1 and Lemmas 10.8, 10.9, 12.3, 12.4, 12.5, 12.7 and 12.9 yields

(12.4) $\quad G = HK$ with H and K α-invariant soluble Hall subgroups of G.

We further observe that

(12.5) $$Z(T) \leqq N_T(K).$$

If $L_iT \neq TL_i$ for $i = 2$ or 3, then, as $Z(T) \leqq T_{\sigma\tau}$ rules out $L_i^* \leqq N_{L_i}(T)$, we have $Z(T) \leqq N_T(L_i)$ by Lemmas 7.6(iv) and 7.7(g). If $L_{23}T \neq TL_{23}$, then we may also deduce that $Z(T) \leqq N_T(L_{23})$. Appealing to Lemma 7.10 we then have (12.5).

(12.6) $\tilde{L}_0^+K = K\tilde{L}_0^+$ is a soluble Hall subgroup of G of odd order.

Lemma 12.5 and 12.9 and (2.4) and (2.5) imply (12.6).

We shall use $\tilde{H}$ to denote the α-invariant Hall $(\pi(\tilde{L}_0^+))'$-subgroup of H.

(12.7) (i) Let F be an α-invariant subgroup of G of odd order with $F \geqq K$. Then $F \leqq N_G(Z(J(K)))$ where $J(K)$ denotes the Thompson subgroup of K; note that $Z(J(K))$ is a non-trivial characteristic subgroup of K (see [1]).

(ii) $G = \tilde{H}N_G(Z(J(K)))$.

(i) Since $F \geqq K$, $F = (F \cap H)K$. Because $(G, \langle\alpha\rangle)$ satisfies Hypothesis III $O_{\xi'}(F) = 1$ where $\xi = \pi(K)$. Applying [Theorem 1; 1] we obtain $Z(J(K)) \trianglelefteq F$ from which (i) follows.

(ii) This follows from (i) and (12.6).

We present the proof in three steps: firstly when T permutes with both L_2 and L_3, secondly when T permutes with only one of L_2 and L_3, and finally when T permutes with neither of L_2 and L_3. In the first two cases we shall require the following result.

(12.8) Suppose $H = TL_1L_i$ where $i = 2$ or 3. Then

(i) $(O_2(TL_i))_\rho = 1$; and

(ii) $O_2(TL_i) = [T, \rho] \trianglelefteq TL_1L_i$.

(i) By Lemma 12.6 $L_i = L_{i_\rho}$ whence, using (2.3)(ix), $T = O_2(TL_i)T_\rho$. Hence $O_2(TL_i) \not\leqq X$ as $T_\rho \leqq X$. Suppose $(O_2(TL_i))_\rho \neq 1$. Then $Z(O_2(TL_i)) \leqq X$ by Lemma 7.10(c). From Lemma 5.1(a) we conclude that either $Z(O_2(TL_i)) \leqq T_{\sigma\tau}$ or $O_2(TL_i) \leqq X$. Thus $Z(O_2(TL_i)) \leqq T_{\sigma\tau}$, whence $[Z(O_2(TL_i)), L_i] = 1$ by (2.3)(xi). Because $1 \neq Z(O_2(TL_i)) \trianglelefteq T$, $Z = Z(T) \cap Z(O_2(TL_i)) \neq 1$. Moreover, as $[Z(T), L_1] = 1$, $Z \leqq Z(TL_1L_i) = Z(\tilde{H})$. Now (12.5) and (12.7)(ii) imply that Z^G is a non-trivial proper α-invariant normal subgroup of G. From this contradiction we deduce that $(O_2(TL_i))_\rho = 1$ must hold.

(ii) Since $L_i = L_{i_\rho}$ we have $[T, \rho] \leqq O_2(TL_i)$ and so, by (2.3)(vii), $[T, \rho] \neq O_2(TL_i)$ would contradict (i). Therefore $[T, \rho] = O_2(TL_1L_i)$. By a wellknown property of soluble groups $O_{\pi_i'}(TL_1L_i) = O_{\pi_i'}(TL_1L_i)O_{\pi_i'}(L_1L_i)$. Since $L_1 = [L_1, \rho] \trianglelefteq L_1L_i$, $O_{\pi_i'}(TL_1L_i) = L_1O_2(TL_i)$ and hence, by (i) and (2.2)(i), $[L_1, O_2(TL_i)] = 1$. Consequently $O_2(TL_i) \trianglelefteq TL_1L_i$ and we have proved (ii).

Case 1. $TL_2 = L_2T$ and $TL_3 = L_3T$

If both of L_2 and L_3 permute with each α-invariant Sylow subgroup of $\tilde{L}_0^-$, then $L_2L_3K = L_2L_3L_{23}\tilde{L}_0^-$ is an α-invariant soluble subgroup of G of odd order. Thus, by (12.4) and (12.7), $G = (TL_1)N_G(Z(J(K)))$. Since $Z(T) \leqq Z(TL_1)$ and $Z(T) \leqq N_G(Z(J(K)))$ we see that $(G, \langle\alpha\rangle)$ cannot satisfy Hypothesis III.

So we may suppose that at least one of L_2 and L_3 does not permute with each of the α-invariant Sylow subgroups of $\tilde{L}_0^-$. In view of Lemma 10.8(iv) we may assume that $L_2U \neq UL_2$ and $L_3U = UL_3$ where U is some α-invariant Sylow subgroup of $\tilde{L}_0^-$. We claim that $L_3 = 1$. Suppose $L_3 \neq 1$. From Lemma 12.6 $L_2 = L_{2_\rho}$ and $L_3 = L_{3_\rho}$. Therefore $U^* = U_\rho \leqq U^*_{\langle\rho\tau\rangle} \leqq \mathcal{P}_U(L_2)$. Since $L_3 \neq L_{3_{\rho\sigma}}$ by Lemma 6.1, we infer that $O_{\pi_3}(L_2L_3) \neq 1$ by (2.13). Now the triple L_2, L_3 and U is at variance with the conclusion of Lemma 5.8(f). Thus $L_3 = 1$, as claimed. So $H = TL_1L_2$.

Using (12.8) we now show that Y permutes with $H = L_1L_2T\tilde{L}_0^+$. For $[T, \rho] \trianglelefteq TY$ and so $(N_G([T, \rho]))_{\{\pi_1, \pi_2, 2, q\}} = L_1L_2TY$. That Y permutes with $\tilde{L}_0^+$ may be shown by using (2.26).

Now $L_2Q \neq QL_2$ by Lemma 12.8 and, since $[L_2, Q_\tau] = 1$ (by Lemma 3.6(ii)), $Z(Q) \leqq N_Q(L_2)$. Since $L_2 = L_{2_\rho}$ and $L_{3_\sigma} = 1$, $Z(Q) \leqq Q_{\rho\sigma}$. So $Z(Q) \leqq Y$. Now $K = L_{23}\tilde{L}_0^-$ admits $\sigma\tau$ fixed-point-freely and so, by (2.10), has Fitting length at most 2. From $Z(Q) \leq Q_{\rho\sigma}$ we may infer that $N_K(Z(Q)) = C_K(Z(Q))$. Hence

$$K = N_K(Q)O_{q'}(K) = C_K(Z(Q))O_{q'}(K).$$

Because $Q \neq Q_\rho = Q^*$, $O_q(K) \neq 1$ by Corollary 4.5, and so $1 \neq Z_1 = Z(Q) \cap O_q(K) \leqq Z(K)$.

Therefore $G = (HY)C_G(Z_1)$. Since $HY \neq G$ and $Z_1 \leqq Y$, Z_1^G is a non-trivial proper α-invariant normal subgroup of G, and with this contradiction case 1 is eliminated.

Case 2. Only one of L_2 and L_3 permutes with T

Since the arguments are symmetric in L_2 and L_3, we will suppose that $L_2T = TL_2$ and $L_3T \neq TL_3$. Note that $\tilde{H} = TL_1L_2$.

If it were the case that L_2 did not permute with Q, then, using (12.8) as in case 1, we obtain $G = (HY)C_G(Z_2)$ where Z_2 is a non-trivial α-invariant subgroup of Y. So we may suppose $L_2Q = QL_2$, and thus L_2 permutes with $\tilde{L}_0^-$ by Lemma 12.8.

We may further suppose that $L_2L_3 \neq L_3L_2$. For $L_2L_3 = L_3L_2$ would imply, using (12.7)(i), that $G = (TL_1)N_G(Z(J(K)))$ which, using (12.5), gives the usual contradiction.

So the situation is as follows: $L_2T = TL_2$, $L_3T \neq TL_3$, $L_2L_3 \neq L_3L_2$ and Q permutes with both L_2 and L_3. We proceed to derive a contradiction from this configuration.

Because $L_2 = L_{2_\rho}$, by Lemma 12.6, $[Q,\rho] \trianglelefteq QL_2$. Since $[Q,\rho] \neq 1$ and $[Q,\rho] \leqq O_q(QL_3)$ we may infer that $O_{\pi_3}(QL_3) \leqq N_{L_3}(L_2) = \mathcal{P}_{L_3}(L_2)$. Now $(N_{L_3}(N_{L_3}(L_2)))^* \leqq N_{L_3}(L_2)$ is not possible by Lemma 4.6, and so, as $L_{3_\rho} \leqq N_{L_3}(L_1)$, we have $N_{L_3}(L_2) \leqq L_{3_\sigma} = L_3^*$. Thus $L_3 = O_{\pi_3}(QL_3)L_3^* = L_{3_\sigma}$.

Clearly $T_\sigma \leqq \mathcal{P}_T(L_3)$. If it were the case that $L_{3_\rho} \leqq \mathcal{P}_{L_3}(T)$, then Lemma 7.7(e) predicts $L_{3_\sigma} \neq L_3$. Hence $T^*_{\langle \rho\sigma \rangle} \leqq \mathcal{P}_T(L_3)$. From the fact that $L_{3_\rho} \leqq N_{L_3}(L_2)$ but $L_{3_\rho} \nleqq \mathcal{P}_{L_3}(T)$ we may assert, using (2.7), that $T = O_2(TL_2)\mathcal{P}_T(L_3)$. Since $T = O_2(TL_2)T_\rho$ and $T_\rho \leqq \mathcal{P}_T(L_3)$ we have $\mathcal{P}_T(L_3) = T_\rho(O_2(TL_2) \cap \mathcal{P}_T(L_3))$.

We will now show that $O_2(TL_2) \cap O_2(L_3\mathcal{P}_T(L_3)) = 1$. Suppose that this does not hold. Then either $Z(O_2(TL_2)) \leqq \mathcal{P}_T(L_3)$ or $O_{\pi_3}(L_3\mathcal{P}_T(L_3)) \leqq \mathcal{P}_{L_3}(T)$. Evidently the former must hold. Because $O_2(TL_2) \nleqq \mathcal{P}_{L_3}(T)$ (otherwise $T = T_\rho O_2(TL_2) \leqq \mathcal{P}_T(L_3)$) we may assert that $Z(O_2(TL_2)) \leqq T_{\sigma\tau}$. As seen previously this gives $1 \neq Z(O_2(TL_2)) \cap Z(T) \leqq Z(TL_1Z_2)$ which, using (12.5) and (12.7), then yields a contradiction. Therefore $O_2(TL_2) \cap O_2(L_3\mathcal{P}_T(L_3)) = 1$.

From $O_2(TL_2) \cap O_2(L_3\mathcal{P}_T(L_3)) = 1$ we may conclude, since $[\mathcal{P}_T(L_3),\sigma] \leqq O_2(L_3\mathcal{P}_T(L_3))$, that $O_2(TL_2) \cap \mathcal{P}_T(L_3) \leqq T_\sigma$. Hence $\mathcal{P}_T(L_3) = T_\rho T_\sigma$ and therefore $[\mathcal{P}_T(L_3),\sigma] \leqq T_\rho$. Since $1 \neq [Z(T_\rho),\sigma\tau] \leqq C_T(O_{\pi_3}(L_3\mathcal{P}_T(L_3)))$, $O_2(L_3\mathcal{P}_T(L_3)) \neq 1$. Consequently $[\mathcal{P}_T(L_3),\sigma] = 1$ would force, by (2.3)(v), $T = T_\sigma$, which is contrary to $TL_3 \neq L_3T$. So $\mathcal{P}_T(L_3) \leqq N_T([\mathcal{P}_T(L_3),\sigma]) \leqq X$ by Lemma 7.10(c). But then $T_\sigma \leqq X$ which Lemma 7.8 shows to be impossible. This is the desired contradiction, and so case 2 is finished.

Case 3. $TL_2 \neq L_2T$ and $TL_3 \neq L_3T$

By (12.7) $G = (TL_1)N_G(Z(J(K)))$ and so, by (12.5), $Z(T)^G$ is a non-trivial proper α-invariant normal subgroup of G, against $(G,\langle\alpha\rangle)$ satisfying Hypothesis III.

The proof of Theorem 12.2 is complete.

13. α-Invariant Sylow subgroups of n-p type I, II, III and IV

In this section we shall be working under the following hypothesis.

HYPOTHESIS 13.1. G possesses α-invariant Sylow subgroups of n-p types I, II, III and IV.

As in the two preceding sections our aim is to demonstrate the following theorem, which, in contrast to the first two cases, is proved by purely local considerations.

THEOREM 13.2. *Hypotheses* 13.1 *and III are incompatible.*

PROOF. Suppose $(G, \langle\alpha\rangle)$ satisfies Hypotheses 13.1 and III.

By Hypothesis 13.1 there exists some α-invariant Sylow q-subgroup Q of G of type Ψ with respect to which T is of n-p type III. We may assume that $T_\rho \leqq \mathcal{P}_T(Q)$ and $Q_\sigma, Q_\tau \leqq \mathcal{P}_Q(T)$, and consequently, by Lemma 7.10, $\mathcal{P}_Q(T) \leqq Q_\rho = Q^* \neq Q$ and $Z(T) = Z(T)_{\sigma\tau}$. Also, let P denote an α-invariant Sylow p-subgroup of G which is of n-p type I. By Lemma 10.5 $P \neq Q$.

Suppose $p \neq 2$. Let U be an α-invariant Sylow u-subgroup of G of type Ψ with respect to which P is of n-p type I. Note that $Q \neq U \neq T$ by Lemma 10.2. If $PQ \neq QP$, then, since $2 \notin \{p, q\}$, P is either of n-p type I or II with respect to Q. Lemmas 10.2 and 10.5 show that neither possibility can occur, and therefore $PQ = QP$. But then Lemma 10.4 applied to P, Q and U predicts that $p = 2$, which is not the case. Hence $p = 2$.

Thus we may suppose that T is the only α-invariant Sylow subgroup of G of n-p type I. So T is of n-p type I with respect to the α-invariant Sylow u-subgroup U (say). So $T^* \leqq N_T(U)$. Since $u \neq 2 \neq q$ neither U nor Q can be of n-p type I or III and thus $UQ = QU$. Now we consider $\mathcal{P}_{QU}(T_\sigma)$. Note that $T_\sigma \mathcal{P}_{QU}(T_\sigma) \neq G$ since $T_\sigma \leqq N_T(U) \neq T$. So $\mathcal{P}_{QU}(T_\sigma) = \mathcal{P}_Q(T_\sigma)\mathcal{P}_U(T_\sigma)$ by (2.26). Clearly $Q_\sigma \leqq \mathcal{P}_Q(T_\sigma)$ and also, since $T_\sigma \leqq N_T(U)$, $\mathcal{P}_U(T_\sigma) = U$. So U normalizes $O_q(QU) \cap \mathcal{P}_Q(T_\sigma) \geqq (O_q(QU))_\sigma$. Whence, by (2.14)(i),

$$O_q(QU) = C_{O_q(QU)}([U, \sigma])(O_q(QU) \cap \mathcal{P}_Q(T_\sigma)).$$

By Lemma 7.8 $[U, \sigma] \neq 1$ and so either $T_\sigma \leqq \mathcal{P}_T(Q)$ or $C_Q([U, \sigma]) \leqq \mathcal{P}_Q(T)$. The former is untenable by Lemma 7.8 and so $C_Q([U, \sigma]) \leqq \mathcal{P}_Q(T)$. Moreover $T_\sigma \not\leqq \mathcal{P}_T(Q)$ implies $\mathcal{P}_Q(T_\sigma) \leqq \mathcal{P}_Q(T)$ and so $O_q(QU) \leqq \mathcal{P}_Q(T)$. But then, using Corollary 4.5,

$$Q = O_q(QU)Q^* = \mathcal{P}_Q(T)Q_\rho = Q_\rho,$$

whereas $Q \neq Q_\rho$.

With this contradiction we have established Theorem 13.2.

Combining Theorems 11.2, 12.2 and 13.2 gives Theorem 10.1.

REFERENCES

1. Z. Arad and G. Glauberman, *A characteristic subgroup of a group of odd order*, Pac. J. Math. **56**(2) (1975), 305–319.

2. D. Gorenstein, *Finite Groups*, Harper and Row, New York, 1968.

3. P. J. Rowley, *Solubility of finite groups admitting a fixed-point-free automorphism of order rst I*, Pac. J. Math. **193** (1981), 201–235.

4. P. J. Rowley, *Solubility of finite groups admitting a fixed-point-free automorphism of order rst II*, J. Algebra **83** (1983), 293–348.